Patterns formed by spiral pairs in oscillatory media

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(Received 21 July 2000; published 19 December 2000)

We investigate by numerical simulations the dynamics of spiral pairs which are found in oscillatory and excitable media as described by the complex Ginzburg-Landau equation. Our simulations include two spiral pairs which approach each other and interact. The spirals typically exchange partners and form new pairs. This type of interaction gives rise to some patterns: a right-angle scattering pattern, a rotating state coming out of two traveling pairs, and a symmetric spiral lattice. The scenarios are compared to other conservative and dissipative systems where scattering at right angles is also observed.

DOI: 10.1103/PhysRevE.63.011103 PACS number(s): 82.40.Ck, 05.70.Ln, 47.32.Cc, 05.45.Yv

Spiral waves are point defects in two dimensions (2D) which are observed in oscillatory and excitable media. Examples of such media are some chemical oscillations [1], thermal convection in binary fluids [2], and nematic liquid crystals [3].

A simple mathematical modeling for oscillatory systems is provided by the complex Ginzburg-Landau equation (CGLE)

$$\frac{\partial A}{\partial t}=A+(1+ib)\Delta A-(1+ic)|A|^2A. \qquad (1)$$

The complex field A describes the amplitude and phase of the modulations of the pattern [1,4]. Here b and c are real parameters and Δ denotes the Laplacian operator. The CGLE plays the role of a normal form in the vicinity of a supercritical bifurcation to an oscillatory state in spatially extended systems and is thus very general.

We are interested in the two-dimensional space where spiral solutions can be found. An isolated spiral is of the form

$$A(\rho,\phi)=F(\rho)\exp\{i[\omega t+\sigma\phi+\psi(\rho)]\},\quad \sigma=\pm 1, \qquad (2)$$

where ρ and ϕ denote the polar coordinates.

The functions F and ψ have the following asymptotic behavior:

$$F(\rho)\to\sqrt{1-k^2},\quad \psi'(\rho)\to k,\qquad \rho\to\infty,$$

$$F(\rho)\sim\rho,\quad \psi'(\rho)\sim\rho,\qquad \rho\to 0. \qquad (3)$$

The phase ϕ changes by $2\pi\sigma$ along a closed orbit around the spiral center where $F=0$. We say that the spirals (2) carry a topological charge σ which can take two values. In order to distinguish between the two cases, we shall call the solution with $\sigma=1$ a ''spiral'' and the one with $\sigma=-1$ an ''antispiral.'' Away from boundaries the topological charge is conserved; i.e., spirals can only disappear by annihilation with an antispiral. The frequency is given by $\omega=-bk^2-c(1-k^2)$ where the constant k is the asymptotic wave number of the waves emitted by the spiral. For given b,c, the wave number k has a unique value which can be determined numerically [5].

In the appropriate parameter range spiral solutions of Eq. (1) are spontaneously formed around a topological defect of the field and are stable with respect to perturbations. The behavior of a system of spirals depends substantially on the values of the parameters of the equation. One of the interesting cases is the formation of a vortex glass whose statics and dynamics has attracted much interest [6].

The states formed by two spirals have also been investigated [7]. For a certain domain of the b-c parameter space, two spirals interact to form a bound state. In particular, a spiral and an antispiral propagate perpendicular to the line connecting their cores. The distance between them remains constant during the motion and the bound state is stable with respect to small changes of this distance. The equilibrium distance and the velocity of the bound pair is determined by the parameters of Eq. (1). The velocity diminishes with increasing equilibrium distance. We also mention that it does not seem possible to find a static two-spiral state in a homogeneous background. On the other hand, two spirals (or antispirals) move around each other. The distance between the two spirals and the velocity of the circular motion are similar to those for the oppositely charged pairs, at least for large separation of the spirals.

Both bound states described in the preceding paragraph are actually unstable with respect to the spontaneous breaking of the symmetry between the two spirals which results in a destruction of the spiral pair [8]. In this case one of the two spirals dominates the space and the other one is shrunk to its core and becomes a so-called ''edge vortex'' (in finite systems).

In all numerical simulations described below we choose $b=0$ and $c>0$ (the latter is not a restriction). According to Ref. [7], bound pairs exist for $c\gtrsim 0.845$. For $c\gtrsim 1.3$ we have the onset of convective instability [9]. As a conclusion, spiral pairs exist and are long lived for $0.845\lesssim c\lesssim 1.3$. All results carry over to nonzero values of b (not too large) by using the similarity transform as discussed in [7].

The situation which has been described above is reminiscent of pairs of topological objects which are predicted to exist in several conservative systems which have topologically nontrivial solutions [10]. For example, the ferromagnet possesses a family of propagating vortex-antivortex pairs with velocities ranging from zero to the velocity of linear excitations. The velocity is inversely proportional to the distance between the vortex and antivortex, at least for widely separated vortices. To complete the picture, two vortices

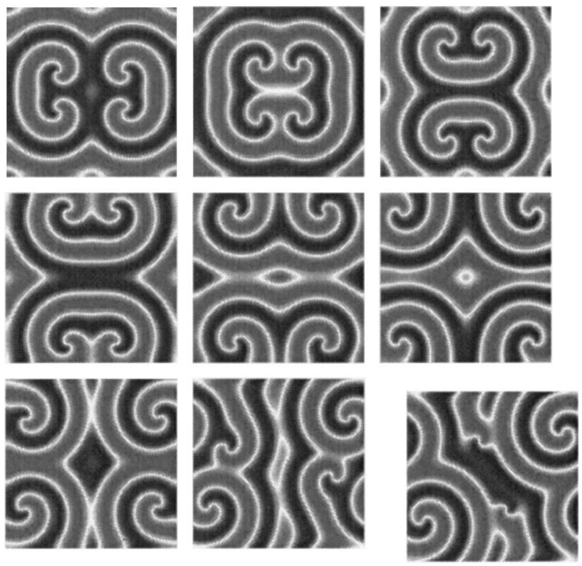

FIG. 1. Time evolution of two spiral pairs, each one being the mirror image of the other. The first exchange of partners occurs between the first and second entries. The symmetry breaking is visible in the seventh entry and is enhanced afterwards. The first eight pictures are equally spaced in time with $\delta t = 700$ time units. The last picture is taken when the system has relaxed, at time $t = 8400$. Time increases from left to right and from top to bottom. Parameter values: $b = 0$, $c = 1.2$. We use periodic boundary conditions and the cell has dimensions 65×65. In this figure as well as in Figs. 3 and 4 we give in a gray-scale code the real part of the field A.

with the same topological charge orbit around each other. This kind of propagating pairs have been termed *solitons* and the study of the interaction between solitons of this kind has led to interesting results. For instance, a head-on collision between two solitons gives a right-angle scattering pattern. This kind of scattering seems to be a characteristic dynamical behavior shared by a variety of conservative systems [11]. Another most striking related case is that of ordinary vortex-antivortex pairs in two-dimensional ideal flows. The formation of bound pairs and their interaction has been investigated in the limit of point vortices [12]. Experiments [13] and numerical studies [14] have been done for some more complicated situations. A right-angle scattering pattern is also here a typical result of the studies.

The interesting results on soliton dynamics have motivated us to investigate the interaction between spiral pairs. The line of reasoning is as follows: since spiral pairs are necessarily traveling, it makes sense to ask what may happen when two pairs move against each other. It is more or less clear that some of the results for the solitons are expected to carry over, at least in a superficial way, to the spiral dynamics.

Our numerical code implements a pseudospectral algorithm with typically 256 modes. We simulate two spiral pairs, as shown in the first entry of Fig. 1. We use the parameter values $b = 0$, $c = 1.2$. The two pairs are initially away from each other; therefore, there is no interaction between them. Each pair quickly relaxes to its equilibrium distance provided that the initial condition is not too far from that. Each pair is the mirror image of the other and they consequently have opposite velocities. We let the system evolve numerically in time and monitor the orbits of the spirals and the antispirals. The pairs move against each other. The spiral and antispiral in the upper half-plane form a new pair and the same happens in the lower half-plane. Subsequently, the two new pairs drift along the vertical axis. The evolution described so far is shown in the entries 1–4 of Fig. 1.

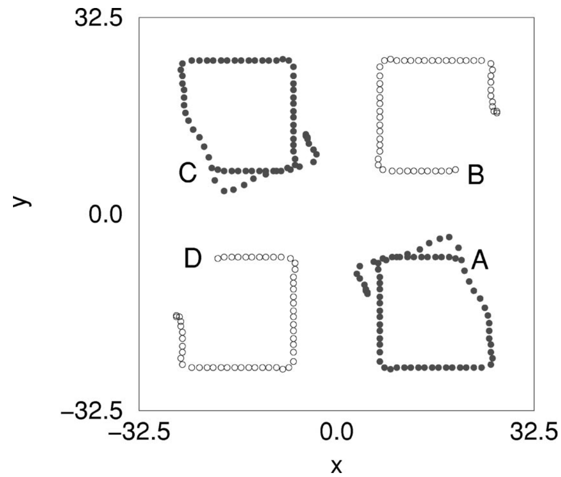

FIG. 2. The orbits of the cores of the spirals of Fig. 1. An open circle denotes the core of a spiral and a solid circle that of an antispiral. The cores are initially located at points *A*, *B*, *C*, and *D*.

The orbits of the cores shown in Fig. 2 help to clarify the picture. We denote by open circles the cores of the spirals and by solid circles those of the antispirals. Let us follow, e.g., the spiral in the upper-right quadrant. It moves initially to the left under the influence of the nearby antispiral. When the pairs approach each other, it feels the influence of the other antispiral, too. Therefore the orbit is curved and the spiral moves in the upward direction. It eventually separates from its initial partner and follows the new one. The curved part of the orbit is quite short, since the interaction between spirals is exponentially decaying with the distance. Therefore the core passes from the influence of the initial partner to the final one within a rather short distance. A careful look in Fig. 2 shows that, during the short time that the four spirals come close together, their cores come shortly closer than the equilibrium distance of a spiral pair.

The time evolution of the system has to respect the symmetries of the equation of motion and of the initial condition. In particular, if one makes the assumption that the spirals and antispirals will not mutually annihilate during interaction, then the output of the above numerical simulation can be considered as the plausible output which respect the present symmetries. However the actual result could only be found numerically.

In the simulations we have used periodic boundary conditions implying image spirals across the boundaries. Thus, further evolution of the simulation gives successive collisions of cores when they approach the boundaries. We would actually expect each of them to circulate around a quadrant

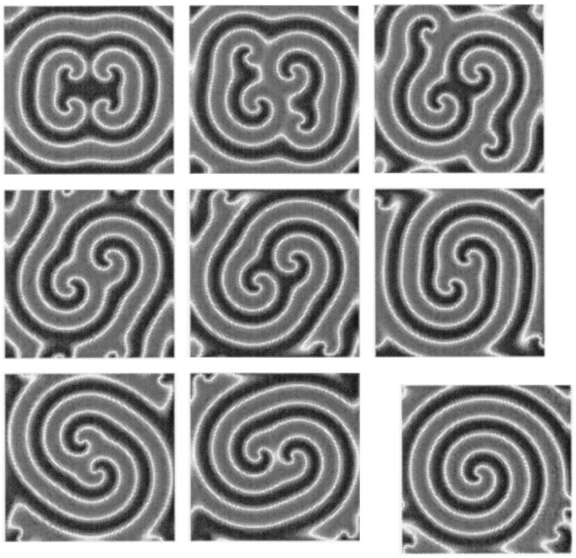

FIG. 3. The initial condition is two pairs shifted in the vertical direction with respect to one another and is given in the first picture. The simulation gives a rotating pair and two ''edge vortices.'' The time interval between the first eight pictures is $\delta t = 1000$. The last picture is taken at time $t = 20\,000$ and represents the final stationary state of the system. Parameter values: $b = 0$, $c = 1.2$. Size of the cell 80×80.

of our simulation space undergoing successive exchanges of partners. The simulation for longer times, which is shown in the entries 4–9 of Fig. 1, results in a spontaneous breaking of the symmetry triggered by unavoidable round-off errors of the computer calculations [8]. After a sufficiently long time, we end up with two almost static spirals which occupy the whole space, while the antispirals have been reduced to their cores (''edge vortices'') and pushed into the region of the shock between the dominating spirals. Such asymmetric lattices of spirals have been studied and found to be stationary and stable.

We are also interested in the time evolution of the cores when the initial condition is slightly perturbed. We start a new simulation with the two spiral pairs as in Fig. 1, but the pairs are now slightly shifted with respect to one another in the vertical direction. This situation is analogous to the collision between particles with a nonzero impact parameter. The results are shown in Fig. 3. When the pairs approach, they form asymmetric spiral-antispiral pairs in the upper and lower half-planes. The asymmetry is enhanced [8] and finally the antispirals degenerate into edge vortices. A rotating bound state of two spirals is formed. This situation is presented in pictures 6–8 of Fig. 3. In the third entry the two antispirals have almost been reduced to their cores. The rotating motion starts at the sixth entry; that is, the spirals need some time, after the destruction of the spiral-antispiral pairs, to organize into the rotating state. The rotating state survives for more than a full rotation and the pair is eventually destroyed due to a new symmetry breaking. The final result is shown in the last entry of Fig. 3. We have a spiral dominating the space and three edge vortices which form a stationary state.

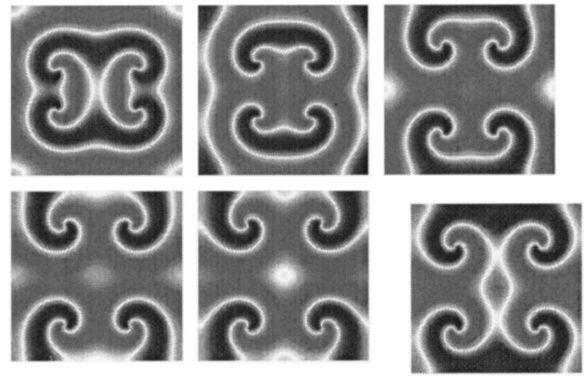

FIG. 4. Two spiral-antispiral pairs are shown in the first entry. The distance between the spirals is chosen sufficiently large so that the defects repel each other. Each defect follows a spiraling orbit in a quadrant. The system relaxes to a symmetric lattice. The first five pictures are equally spaced in time with $\delta t = 2500$ time units. The last picture is taken when the system has almost relaxed at $t = 50\,000$. Parameter values: $b = 0$, $c = 0.7$. Size of the cell 100×100.

The exact time evolution depends on the initial condition. In many simulations the two dominating spirals stay somewhat away from each other and the velocity of rotation is consequently much lower than that shown in Fig. 3. Then the scenario of Fig. 3 takes much longer time.

The scattering scenarios that have been described up to now occur for the values of the parameters b, c where bound pairs of spirals exist and the emitted waves are convectively stable. From the similarity transform one sees that this is the case between the curve $(c-b)/(1+bc) \simeq 0.845$ and the convective stability limit (see Fig. 6 of [7]).

The present numerical results can be compared to those for a ferromagnet. In an analogous numerical simulation in the conservative system, one obtains the formation of a rotating two-vortex state. However, this state exists only for a short time. It is then resolved in favor of the formation of vortex-antivortex pairs which finally drift away from each other.

We now move to a different region of the b-c parameter space where no bound pairs of spirals exist; i.e., we choose $b = 0$ and $0 < c \lesssim 0.845$. An isolated spiral is still a stable object for these parameter values and a spiral and an antispiral are moving due to their interaction. Their velocities have a tangential component, which is equal for the two spirals. It also has a radial component, along the line connecting their centers. This is repulsive for not too small separation between the spirals. In contrast to the case of the bound spiral pairs, we expect here no spontaneous symmetry breaking. On the contrary, any asymmetry of the spiral pair will be suppressed [8].

We choose the parameter values $b = 0$, $c = 0.7$. The results of the simulation are shown in Fig. 4. The evolution of the structure, at its initial stage, resembles that of Fig. 1. However, in the present case the spiral and the antispiral of a single pair repel each other. One can roughly imagine the orbit of each spiral, taking into account the interaction of each one with the others as well as with the spirals implied

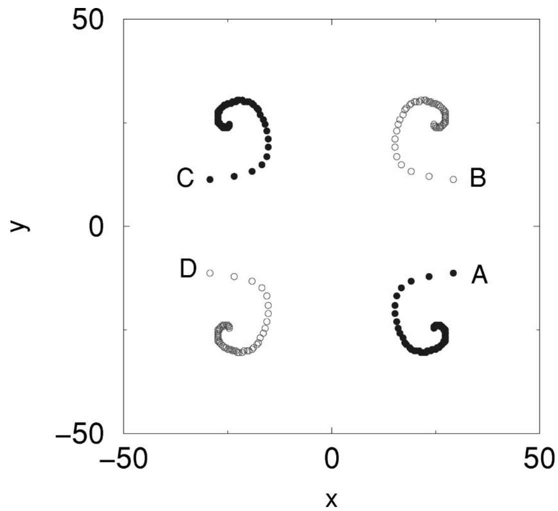

FIG. 5. The orbits of the cores of the spirals of Fig. 4. An open circle denotes the core of a spiral and a solid circle that of an antispiral. The cores are initially located at points *A*, *B*, *C*, and *D*.

by the periodic boundary conditions. The orbits followed by the four spirals are depicted in Fig. 5. Each defect follows a spiraling orbit and ends up in the center of the corresponding quadrant. The periodic boundary conditions imply that the final output is a symmetric spiral lattice. The symmetric lattice is, according to [8] a stationary solution of Eq. (1) and it is stable with respect to nonperiodic perturbations. It comes up here in a natural way through the interaction process of two spiral pairs.

One may also perform simulations similar to the previous one, but with an asymmetric initial condition. The asymmetry may be inserted in the initial conditions in a variety of ways. In all the cases that we have investigated the system relaxes to the symmetric spiral lattice state provided that the initial asymmetry is not too large.

The behavior described in the last paragraphs occurs generically in the region of the (b-c) plane where spirals are stable and no bound pairs exist. The region is confined by the two branches of the curve $(c-b)/(1+bc)\simeq 0.845$ [7].

Let us now compare the results with the dynamics of spots which have been studied in reaction-diffusion (RD) systems [15–17]. Two moving spots may collide head-on and give two different results. In particular, in the model of Ref. [15], it is found that two slow spots are ''elastically'' scattered. This means that they come to a minimum distance and then bounce back. However, for larger velocities, the two spots are fused together and then reappear moving on a direction perpendicular to the initial direction of motion. Analogous results are found for a different RD system [16].

The scattering of spots can be compared to our results for the scattering of pairs of spirals. We may claim that a right-angle scattering behavior has been established in both system (1) and the spots in RD systems. However, contrary to the scattering of spots, an effective bouncing back of colliding spiral pairs does not seem possible.

On the side of conservative systems, the scattering of solitons in relativistic systems has been studied for long [18]. One expects that a right-angle scattering will appear generically in two-dimensional models which have soliton solutions. This includes solitons of the vortex-antivortex type as well as solitons which obey relativistic dynamics [11].

Recently, collisions for a class of so-called nontopological solitons have been studied in models applying to cosmology [19]. Here, in addition to passing through each other and a scattering at right angles, one can also have a combination of the two cases. That is, the two initial colliding solitons may split during the collision to give four new ones. This possibility has some similarities with the replication of spots observed in [20]. Here, the substantial difference is that spots may replicate while solitons may at most split in two ''halves.''

In conclusion, we have taken here advantage of the dynamical properties of spiral pairs and have used them in numerical simulations. We investigate the time evolution of the system of two spiral pairs as well as the patterns that are formed. The results depend not only on the parameters of the CGLE (1) but also on the initial conditions of the simulation. We have shown numerically that two identical spiral pairs may combine to give a right-angle scattering pattern, resembling the soliton scattering. Another possibility is the formation of a rotating spiral pair from two initial traveling spiral-antispiral pairs. One more possibility is the symmetric lattice of spirals obtained in the appropriate parameter range. We observe that the dynamical behavior of spirals have some similarities to the dynamics of vortex-antivortex pairs in conservative systems. Furthermore, the scattering behavior of spots in reaction-diffusion systems bears similarities to that of solitons in relativistic models.

S.K. and L.K. thank the Max-Planck Institute für Physik Komplexer Systeme (Dresden, Germany), where part of this work was done, for its hospitality. We also acknowledge support from the TMR program of the EU ''patterns, noise and chaos'' under Contract No. ERBFMRXCT-960085.

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